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Stability of Functional Differential Equations with Perturbed Lags

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INTRODUCTION

This paper is concerned with stability under perturbations of functional differential equations. The type of perturbations considered are general enough to include perturbations in the arguments of the dependent variable. The method of investigation uses Lyapunov functionals to obtain scalar differential equalities whose solutions dominate the solutions of the perturbed equation.

NOTATIONS AND BASIC DEFINITIONS

In discussing functional differential equations, we will use the standard notation introduced by Hale; see [2]. Given a function $x: [-q, A] \to R^n$, A, $q > 0$, for $0 \leqslant t < A$, denote by x_t the function x restricted to $[t - q, t]$, that is, for $0 \leqslant t < A$, $-q \leqslant \theta \leqslant 0$, $x_t(\theta) = x(t + \theta)$.

By a functional differential equation is meant an equation of the form

$$\dot{x}(t) = f(t, x_t). \tag{1}$$

Here f is a mapping from $R \times C[-q, 0]$ into R^n, where $C[-q, 0]$ is the space of continuous functions from $[-q, 0]$ to R^n. The norm on $C[-q, 0]$ is given by $\|\phi\| = \sup_{-q\leqslant\phi\leqslant 0} |\phi(\theta)|$, where $|\phi(\theta)|$ is a vector norm in R^n. Standard existence and uniqueness theorems apply to (1); see [2] for details.

On occasion we will need to consider the restriction of x to the interval $[t - 2q, t]$, rather than $[t - q, t]$. (For x defined on $[-q, A)$, this is meaningful only for $t \geqslant q$.) Accordingly, we will also consider the space $C[-2q, 0]$ of continuous functions from $[-2q, 0]$ to R^n. The norm on $C[-2q, 0]$ is given by $\|\phi\|_{2q} = \sup_{-2q\leqslant\theta\leqslant 0} |\phi(\theta)|$. For $\phi \in C[-q, 0]$, imbed ϕ into $C[-2q, 0]$ by some trivial extension so that $\|\phi\| = \|\phi\|_{2q}$, (e.g., $\phi(\theta) = \phi(-q)$, for $-2q \leqslant \theta < -q$).

By a functional V is meant a map from $R \times C \to R$. Given a functional V, define $\dot{V}_1(t, \phi)$, the derivative of V with respect to Eq. (1), as follows: Let $x_t(t_0, \phi)$ denote the solution of (1), defined for $t \geqslant t_0$, with the initial condition $x_{t_0}(t_0, \phi) = \phi$. Then

$$\dot{V}_1(t, \phi) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)].$$

$\dot{V}_1(t, \phi)$ is the upper right-hand derivative of $V(t, \phi)$ along solutions of (1).

1. Illustration of the Basic Notion

Consider the equation

$$\dot{y}(t) = L(y_t), \tag{1.1}$$

where L is a bounded linear map from $C[-q, 0]$ into R^n, $\| L \| \leqslant H$. Let $y_t(t_0, \phi)$ denote the solution of (1.1) through ϕ at $t = t_0$.

Suppose the origin is exponentially stable in (1.1), that is, for $t \geqslant t_0$, and all ϕ, we have

$$\| y_t(t_0, \phi) \| \leqslant K e^{-\alpha(t-t_0)} \| \phi \|, \qquad \alpha > 0, \quad K \geqslant 1. \tag{1.2}$$

As is well known, there is a converse theorem due to Hale and Yoshizawa which asserts the existence of a functional $V(\phi)$ linear in ϕ satisfying

$$\begin{aligned} &\text{(a)} \quad \| \phi \| \leqslant V(\phi) \leqslant K \| \phi \|, \\ &\text{(b)} \quad \dot{V}_{1.1}(\phi) \leqslant -\alpha V(t, \phi). \end{aligned} \tag{1.3}$$

Clearly, solving the inequality (1.3b), and using the bounds in (1.3a), one can obtain the estimation in (1.2), so (1.2) and (1.3) are equivalent.

Hale goes on to consider the perturbed equation

$$\dot{y}(t) = L(y_t) + Y(t, y_t), \tag{1.4}$$

when Y is some continuous map from $R \times C[-q, 0]$ into R^n. It can be readily shown, using the fact that V is linear in ϕ, therefore Lipschitzian, that V satisfies

$$\dot{V}_{1.4} \leqslant -\alpha V + K \mid Y(t, y_t) \mid. \tag{1.5}$$

If we make some suitable assumption about Y, e.g., $\mid Y(t, \phi) \mid \leqslant G(t, \| \phi \|)$, with $G(t, r)$ nondecreasing in r, then we obtain, using (1.5) and (1.3a),

$$\dot{V}_{1.4} \leqslant -\alpha V + KG(t, V), \tag{1.6}$$

and again by (1.3a), if u is a maximal solution of the differential inequality (1.6), $\| y_t(t_0, \phi)\| \leqslant u(t)$, where now y_t is a solution of (1.4).

This is all well and good if Y possesses a bound G with suitable properties with respect to the C^0-norm on $[-q, 0]$. But if the equation we wish to analyze is $\dot{y}(t) = -ay(t) + by(t - r - \tau(y_t))$, for example, it is natural to let $L(\phi) = -a\phi(0) + b\phi(-r)$, and $Y(\phi) = b[\phi(-r - \tau(\phi)) - \phi(-r)]$. The appropriate bound on Y in this case is

$$| Y(\phi)| \leqslant | b | \cdot \| \dot{\phi} \| \cdot | \tau(\phi)| , \tag{1.7}$$

which now involves the C^1 norm of ϕ on $[-q, 0]$.

The purpose of this paper is to modify the functional $V(\phi)$ in such a manner so that the modified functional W can be shown to satisfy a differential inequality similar to (1.6), but now reflecting the C^1-bound on Y.

The procedure is as follows. The estimate (1.2) on the norm of y_t can be extended to an estimate of $\| y_t \|_{2q}$, the sup norm over $[-2q, 0]$. This is evident as

$$\begin{aligned} \| y_t(t_0, \phi)\|_{2q} &= \sup_{-q \leqslant \theta \leqslant 0} \| y_{t+\theta}(t_0, \phi)\| \\ &\leqslant \sup_{\theta} K e^{-\alpha(t+\theta-t_0)} \| \phi \| \\ &\leqslant K e^{\alpha q} e^{-\alpha(t-t_0)} \| \phi \| \\ &\leqslant K_1 e^{-\alpha(t-t_0)} \| \phi \|_{2q}, \qquad \text{where} \qquad K_1 = K e^{\alpha q}. \end{aligned} \tag{1.8}$$

Now define $W(\phi)$, using (1.8) and $\| y_t \|_{2q}$, so that W satisfies (1.3), with $\| \phi \|$ replaced by $\| \phi \|_{2q}$ throughout, and K_1 replacing K.

Again it is easy to show that W satisfies

$$\dot{W}_{1.4} \leqslant -\alpha W + K_1 | Y(t, \phi)| .$$

We will first use a C^0-estimate for Y, in our example, namely,

$$| Y(t, \phi)| \leqslant 2b \| \phi \| .$$

Thus, on the interval $[t_0, t_0 + q]$, W satisfies

$$\dot{W}_{1.4} \leqslant -\alpha W + 2bW. \tag{1.9}$$

Now, for $t \geqslant t_0 + q$, note that

$$| \dot{y}(t + \theta)| \leqslant H \| y_{t+\theta} \| + 2 | b | \| y_{t+\theta} \| \leqslant (H + 2 | b |) \| y_t \|_{2q} .$$

So, for $t \geqslant t_0 + q$,

$$\begin{aligned} | Y(t, y_t)| &\leqslant | b | \, | \dot{y}_t | \cdot | \tau(y_t)| \\ &\leqslant | b | (H + 2 | b |) \| y_t \|_{2q} | \tau(y_t)| , \end{aligned}$$

and for $t \geqslant t_0 + q$, W satisfies

$$\dot{W}_{1.4} \leqslant -\alpha W + K_1 | b | (H + 2 | b |) | \tau(y_t)| W.$$

If we now assume $| \tau(\phi)|$ is dominated by some function monotonic in $\| \phi \|$, e.g., $| \tau(\phi)| \leqslant g(\| \phi \|)$, we obtain

$$\dot{W}_{1.4} \leqslant (-\alpha + K_1 | b | (H + 2 | b |) g(W)) W, \tag{1.10}$$

and by suitable hypotheses on $g(W)$, we can obtain stability results concerning solutions of (1.4).

Of course, inequality (1.9) applies on the initial lag interval, but this is linear in W, and the solution at $t_0 + q$ can clearly be expressed in terms of the initial conditions at t_0. So the controlling inequality is (1.10), and solutions of this scalar differential inequality dominate solutions of (1.4), even though the perturbation Y is not necessarily small in the C^0 norm, but depends in an intrinsic manner on the C^1-norm.

A typical example of the results obtainable in this fashion is to assume that $| \tau(\phi)| = O(\| \phi \|)$ as $| \phi | \to 0$. Then for W sufficiently small, (1.10) reduces to $\dot{W}_{1.4} \leqslant -(\alpha/2) W$, and the origin in (1.4) is locally exponentially stable.

The remaining portion of this paper consists of various generalizations of the above, generalizing the type of estimate assumed, the type of perturbation Y allowed, and the unperturbed equation need not be linear. Section 4 deals with applications to further illustrate the theory.

2. The Unperturbed Equation

Consider the equation

$$\dot{y}(t) = L(t, y_t), \tag{2.1}$$

where $L(t, \phi)$ maps $R \times C[-q, 0]$ into R^n, L is continuous in t, linear in ϕ and $| L(t, \phi)| \leqslant H \| \phi \|$, for some constant $H > 0$, and all $t > 0$. Let $y_t(t_0, \phi)$ denote the solution of (1) through ϕ at $t = t_0$.

Assume that for all t, t_0, $t \geqslant t_0 \geqslant 0$, and all $\phi \in C[-q, 0]$, $y_t(t_0, \phi)$ satisfies

$$\| y_t(t_0, \phi)\| \leqslant K(t_0) \, e^{-[\alpha(t) - \alpha(t_0)]} \| \phi \| , \tag{2.2}$$

where K is continuous and positive for $t \geqslant 0$, and α has a continuous derivative for $t \geqslant 0$.

The following lemma is well known, [1].

LEMMA 2.1. *Assume the solutions of* (2.1) *satisfy* (2.2). *Then there is a functional* $V(t, \phi)$, *continuous for* $t \geqslant 0$, $\phi \in C[-q, 0]$, *which satisfies the following*

$$\begin{aligned} &\text{(a)} \quad \|\phi\| \leqslant V(t, \phi) \leqslant K(t)\|\phi\|, \\ &\text{(b)} \quad \dot{V}_{2.1}(t, \phi) \leqslant -\dot{\alpha}(t) V(t, \phi), \\ &\text{(c)} \quad |V(t, \phi_1) - V(t, \phi_2)| \leqslant K(t)\|\phi_1 - \phi_2\|. \end{aligned} \tag{2.3}$$

As in Section 1, it is easy to establish:

LEMMA 2.2. *Assume* α *satisfies at least one of the following, for* $t \geqslant 0$: (i) $|\dot{\alpha}(t)| \leqslant C$, (ii) $\dot{\alpha}(t) \leqslant 0$, *or* (iii) $\dot{\alpha}(t) \geqslant 0$. *Then* $y_t(t_0, \phi)$ *satisfies*

$$\|y_t(t_0, \phi)\|_{2q} \leqslant K_1(t_0)\, e^{-(\beta(t)-\beta(t_0))}\|\phi\|_{2q}, \tag{2.4}$$

where under condition (i), $K_1(t_0) = e^{Cq}K(t_0)$, $\beta = \alpha$; *if* (ii) *is valid*, $K_1 = K$, $\beta = \alpha$, *and if* (iii) *is satisfied*, $\beta(t) = \alpha(t - q)$.

$$K_1(t_0) = K(t_0)\, e^{\alpha(t_0)-\alpha(t_0-q)}.$$

Using estimate (2.4), the same arguments used to establish Lemma 2.1 imply the following lemma.

LEMMA 2.3. *Assume the solutions of* (2.1) *satisfy* (2.4). *Then there is a functional* $W(t, \phi)$, *continuous for* $t \geqslant 0$, $\phi \in C[-q, 0]$, *which satisfies the following.*

$$\begin{aligned} &\text{(a)} \quad \|\phi\|_{2q} \leqslant W(t, \phi) \leqslant K_1(t)\|\phi\|_{2q}, \\ &\text{(b)} \quad \dot{W}_{2.1}(t, \phi) \leqslant -\dot{\beta}(t) W(t, \phi), \\ &\text{(c)} \quad |W(t, \phi_1) - W(t, \phi_2)| \leqslant K_1(t)\|\phi_1 - \phi_2\|_{2q}. \end{aligned} \tag{2.5}$$

With only weak restrictions on $\alpha(t)$ and $K(t)$, a lemma similar to Lemma 2.1 also appears in Hale [1] which applies to nonlinear systems, namely, the following.

LEMMA 2.4. *Consider the system*

$$\dot{x}(t) = g(t, x_t), \tag{2.6}$$

g *continuous in* (t, ϕ) *and Lipschitzian in* ϕ, *for* $t \geqslant 0$, $\|\phi\| \leqslant B$, *whose solutions satisfy* (2.2). *If* $\alpha(t)$ *is nondecreasing*, $K(t)$ *bounded, and if, for some* k, $0 < k < 1$, *there exists a* $T > 0$ *such that for all* $t \geqslant 0$

$$K(t)\, e^{-k[\alpha(t+T)-\alpha(t)]} \leqslant 1,$$

then there exists a function $V(t, \phi)$, continuous for $t \geqslant 0$, $\| \phi \| \leqslant B$, such that

$$\begin{aligned} &\text{(a)} \quad \| \phi \| \leqslant V(t, \phi) \leqslant K(t) \| \phi \| , \\ &\text{(b)} \quad \dot{V}_{2.6}(t, \phi) \leqslant -(1 - k) \dot{\alpha}(t) V(t, \phi), \qquad (2.7) \\ &\text{(c)} \quad \| V(t, \phi_1) - V(t, \phi_2) \| \leqslant e^{LT} \sup_{0 \leqslant \tau \leqslant T} \exp(\alpha(t + \tau) - \alpha(t))) \cdot \| \phi_1 - \phi_2 \| . \end{aligned}$$

Here L is a Lipschitz constant for g.

Just as Lemma 2.1 has its nonlinear counterpart in Lemma 2.4, so does Lemma 2.3. We may state the following.

LEMMA 2.5. *Assume solutions of* (2.6) *satisfy* (2.2), *and let g, α and K satisfy the conditions of Lemma* 2.4. *Then there exists a functional $W(t, \phi)$ continuous in t, ϕ, for $t \geqslant 0$, $\| \phi \| \leqslant B$ such that*

$$\begin{aligned} &\text{(a)} \quad \| \phi \|_{2q} \leqslant W(t, \phi) \leqslant K_1(t) \| \phi \|_{2q} , \\ &\text{(b)} \quad \dot{W}_{2.6}(t, \phi) \leqslant -(1 - k) \dot{\beta}(t) W(t, q), \qquad (2.8) \\ &\text{(c)} \quad | W(t, \phi_1) - W(t, \phi_2) | \leqslant K_2(t) \| \phi_1 - \phi_2 \|_{2q} , \end{aligned}$$

where

$$K_1(t) = K(t) e^{\alpha(t) - \alpha(t-q)}, \qquad \beta(t) = \alpha(t - q),$$

and

$$K_2(t) = e^{LT} \sup_{0 \leqslant \tau \leqslant T} e^{(1-k)\alpha(t-q+\tau) - \alpha(t-q)}.$$

Here we must assume

$$K_1(t) e^{-k[\beta(t+T) - \beta(t)]} \leqslant 1.$$

Proof. Equation (2.2) and $\dot{\alpha}(t) \geqslant 0$ imply by Lemma 2.2 that (2.4) is valid with K_1, β as given above. Then Lemma 2.4 applied to (2.4) gives Lemma 2.5.

3. THE PERTURBED EQUATION

Now consider the perturbed equation

$$\dot{z}(t) = L(t, z_t) + Y(t, z_t), \qquad (3.1)$$

where $Y(t, \phi)$ is continuous for $t \geqslant 0$, $\phi \in C[-q, 0]$, and L is as in (2.1). The following lemma is also in [1].

LEMMA 3.1. *Assume that solutions of* (2.1) *satisfy* (2.2), *and let $V(t, \phi)$ be the functional defined in Lemma* 2.1. *Then*

$$\dot{V}_{3.1}(t, \phi) \leqslant -\dot{\alpha}(t) V(t, \phi) + K(t) | Y(t, \phi) | , \qquad \textit{for } t \geqslant 0, \quad \phi \in C[-q, 0].$$

From Lemmas 2.2, 2.3, and 3.1, we readily obtain the following.

LEMMA 3.2. *Assume that the solutions of* (2.1) *satisfy* (2.2), *and assume* α *satisfies one of the hypotheses of Lemma* 2.2. *Let* W *be the functional defined in Lemma* 2.3. *Then*

$$\dot{W}_{3.1}(t,\phi) \leqslant -\dot{\beta}(t)\,W(t,\phi) + K_1(t)\,|\,Y(t,\phi)|\,.$$

Lemma 3.1 also may be applied to the nonlinear problem.

LEMMA 3.3 *Consider the perturbed equation*

$$\dot{z}(t) = g(t, z_t) + Y(t, z_t), \tag{3.2}$$

where g *and* Y *are continuous in* (t,ϕ), *Lipschitzian in* ϕ *for* $t \geqslant 0$, $\|\phi\| \leqslant B$.
Assume that solutions of (2.6) *satisfy* (2.2), *and assume that the conditions of Lemma* 2.5 *are valid. Then*

$$\dot{W}_{3.2}(t,\phi) \leqslant -(1-k)\,\dot{\beta}(t)\,W(t,\phi) + K_2(t)\,Y(t,\phi),$$

where $W(t,\phi)$, β *and* K_2 *are as defined in Lemma* 2.5.

The following two assumptions on $Y(t,\phi)$ will be referred to subsequently as the basic hypothesis.
Assume

$$|\,Y(t,\phi)| \leqslant M(t, \|\phi\|), \qquad \text{for } t \geqslant 0, \quad \|\phi\| \leqslant B, \tag{3.3}$$

and, if $\phi \in C[-q, 0]$ has a continous derivative $\dot{\phi}$, assume

$$|\,Y(t,\phi)| \leqslant N(t, \|\phi\|, \|\dot{\phi}\|), \qquad \text{for } t \geqslant 0, \qquad \|\phi\| \leqslant B. \tag{3.4}$$

Here $M(t,r)$, $N(t,r,s)$ are continuous in all arguments, M is nondecreasing in t and r, independently, and N is nondecreasing in r and s independently. In addition, assume $M(t,0) = 0$ for $t \geqslant 0$.
Then we may state the following.

THEOREM 3.4. *Assume solutions of* (2.1) *satisfy* (2.2), *and* $\alpha(t)$ *satisfies one of the conditions of Lemma* 2.2. *Assume* Y *satisfies the basic Hypotheses* 3.3 *and* 3.4. *Let* $z_t(t,\phi)$ *denote a solution of* (3.1) *with initial condition* ϕ *at* $t = t_0$. *Let* $u(t)$ *be the maximal solution of:*

$$\dot{u} = -\dot{\beta}(t)\,u + K_1(t)\,M(t,u), \qquad \text{for } t_0 \leqslant t \leqslant t_0 + q \tag{3.5}$$

and

$$\dot{u} = -\dot{\beta}(t)\,u + K_1(t)\,N(t, u, Hu + M(t,u)), \qquad \text{for } t \geqslant t_0 + q, \tag{3.6}$$

with $u(t_0) = W(t_0, \phi)$. *Then for* $t \geqslant t_0$, *and wherever* $u(t)$ *is defined, we have:*

$$\| z_t(t_0, \phi) \| \leqslant u(t). \tag{3.7}$$

Proof. From Lemma 3.2, it is clear that on $[t_0, t_0 + q]$, 3.3 implies

$$\dot{W}_{3.1}(t, \phi) \leqslant -\dot{\beta}(t)\, W(t, \phi) + K_1(t)\, M(t, \phi), \tag{3.8}$$

using 2.5a. For $t \geqslant t_0 + q$, (3.1) and (3.3) imply

$$| \dot{z}(t + \theta)| \leqslant H \| z_{t+\theta}(t_0, \phi)\| + M(t + \theta, \| z_{t+\theta}(t_0, \phi)\|),$$

so that

$$\| \dot{z}_t \| \leqslant H \| z_t \|_{2q} + M(t, \| z_t \|_{2q}), \qquad t \geqslant t_0 + q. \tag{3.9}$$

And for $t \geqslant t_0 + q$, z_t has a continuous derivative $\dot{z}_t$, so (3.4) and (3.9) imply

$$| Y(t, z_t)| \leqslant N(t, \| z_t \|, H \| z_t \|_{2q} + M(t, \| z_t \|)). \tag{3.10}$$

By Lemma 3.2, it follows that for $t \geqslant t_0 + q$,

$$\dot{W}_{3.1}(t, \phi) \leqslant -\dot{\beta}(t)\, W(t, \phi) + K_1(t)\, N(t, W(t, \phi), HW(t, \phi) + M(t, W(t, \phi))), \tag{3.11}$$

using (3.10), and (2.5a) again.

From standard arguments, (see [3]), it follows that as $W(t, z_t)$ satisfies (3.8) and (3.11), then $W(t, z_t) \leqslant u(t)$ wherever $u(t)$ is defined. But again by (2.5a), $\| z_t \| \leqslant W(t, z_t)$, and (3.7) follows. This completes the proof of Theorem 1.

Remark 3.5. Observe that in (3.5), $u = 0$ is a solution. So by the usual continuity theorems (see [3]), given t_0, q, $u(t)$ is defined on the interval $[t_0, t_0 + q]$, and $u(t_0 + q)$ can be made as small as desired, provided only that $u(t_0)$ is sufficiently small. Thus (3.6) is the equation which governs solutions of (3.1) for large t.

Theorem 3.4 also has a nonlinear version.

THEOREM 3.6. *Consider the perturbed equation* (3.2), *and assume Lemma* 3.3 *is valid. Assume* $Y(t, \phi)$ *satisfies the basic Hypotheses* 3.3 *and* 3.4.

Assume

$$| g(t, \phi)| \leqslant G(t, \| \phi \|), \qquad t \geqslant 0, \quad \| \phi \| \leqslant B, \tag{3.12}$$

where $G(t, r)$ *is continuous and monotonic in both variables. Let* $z_t(t_0, \phi)$ *denote a solution of* (3.2) *through* ϕ *at* $t = 0$. *Let* $u(t)$ *be a maximal solution of*

$$\dot{u} = -(1 - k)\, \dot{\beta}(t)\, u + K_2(t)\, M(t, u), \qquad \text{for } t_0 \leqslant t \leqslant t_0 + q, \tag{3.13}$$

and

$$\dot{u} = -(1-k)\,\dot{\beta}(t)\,u + K_2(t)\,N(t, u, G(t, u) + M(t, u)), \qquad \textit{for } t \geqslant t_0 + q, \tag{3.14}$$

with

$$u(t_0) = W(t_0, \phi).$$

Then for $t \geqslant t_0$, and wherever $u(t)$ is defined, we have

$$\| z_t(t_0, \phi)\| \leqslant u(t). \tag{3.15}$$

The proof of Theorem 3.6 proceeds exactly as that of Theorem 3.4 and will not be given. Note also that Remark 3.5 applies to (3.13) also, so Eq. (3.14) governs solutions of (3.2) for large t.

4. Applications

The purpose of this section is to illustrate the use of Theorem 3.4, for as stated, it is somewhat general in form. The applications of Theorem 3.6 are similar, and will be discussed only briefly.

In order to apply Theorem 3.4, one must be able to provide estimates on the behavior, for large t, of solutions of (3.6).

$$\dot{u} = -\dot{\beta}(t)\,u + K_1(t)\,N(t, u, Hu + M(t, u)).$$

Here M, N bound the perturbation Y in the C^0, C^1 norm respectively, and $\dot{\beta}$, K_1 reflect the stability properties of the unperturbed equation.

The following two simplifying assumptions include the most important types of stability, and certainly the types most usually present. These assumptions are made only to simplify the following. Equation (3.6) can be analyzed under much more general hypotheses; see for example [1].

$$\beta(t) = \sigma t, \qquad \sigma > 0,\ K_1(t) \text{ a constant}, \qquad \text{for all } t \geqslant 0. \tag{4.1}$$

$$\beta(t) \equiv 0, \qquad \text{and} \qquad K_1 \text{ a constant}, \qquad \text{for all } t \geqslant 0. \tag{4.2}$$

Of course 4.1 is equivalent to assuming (2.1) is exponentially asymptotically stable, and 4.2 is equivalent to assuming 2.1 is uniformly stable.

M, the C^0-bound on Y is not as significant as the C^1-bound N, so for simplicity, let us assume M is independent of t, that is,

$$| Y(t, \phi)| \leqslant m(\| \phi \|), \qquad \text{for } t \geqslant 0, \quad \phi \in C[-q, 0], \quad \| \phi \| \leqslant B, \tag{4.3}$$

where $m(r)$ is continuous and nondecreasing in r, $m(0) = 0$.

With the above assumptions, the dominating equation (3.6) reduces to one of two types.

If (4.1) and (4.3) are assumed, (3.6) becomes

$$\dot{u} = -\sigma u + K_1 N(t, u, Hu + m(u)), \tag{4.4}$$

or, if (4.2) and (4.3) are assumed, we have

$$\dot{u} = K_1 N(t, u, Hu + m(u)). \tag{4.5}$$

To consider (4.4) first, a natural question to ask, since the unperturbed system is exponentially stable, is "what conditions on N and m imply $u \to 0$ as $t \to \infty$?".

One simple condition is the following well-known result.

THEOREM 4.1. *Assume* 4.1 *and* 4.3 *are satisfied. Suppose there exists a continuous function* $\psi(t)$, $t \geqslant 0$, *such that*

$$N(t, u, Hu + m(u)) \leqslant \psi(t) \mid u \mid, \qquad \text{for } t \geqslant 0, \quad \mid u \mid \leqslant D. \tag{4.6}$$

Then, if there exists positive constants T, T_0, *and* γ, $\gamma < 1$, *such that*

$$\frac{1}{T} \int_t^{t+T} \psi(s)\, ds \leqslant \frac{\gamma\sigma}{K_1} \qquad \text{for } t \geqslant T_0, \tag{4.7}$$

then Eq. (4.4) *is exponentially stable at* $u = 0$.

Remark 4.2. Condition 4.7 includes the following

$$N(t, u, Hu + m(u)) = o(\mid u \mid) \qquad \text{as} \qquad \mid u \mid \to 0, \tag{4.8}$$

uniformly in t for $t \geqslant 0$, for we may take $\psi(t) = \epsilon$, for a suitable small constant ϵ, in some D-neighborhood of $u = 0$.

$$\psi(t) \to 0 \qquad \text{as} \qquad t \to \infty, \tag{4.9}$$

(4.7) is valid for T_0 sufficiently large, and $T = 1$.

$$\int_0^\infty \psi(t)\, dt < \infty, \tag{4.10}$$

for again (4.7) is valid for T_0 sufficiently large, and $T = 1$.

Now the next question is: What types of perturbations Y will provide estimates satisfying (4.6)? A simple class of examples is furnished by the following.

$$\dot{y}(t) = \sum_{i=1}^{k} A_i(t)\, y(t - r_i(t)), \qquad \text{(Unperturbed Equation)} \tag{4.11}$$

where the origin is exponentially stable, $r_i(t) \in [0, r]$, r_i and A_i are continuous, $A_i(t)$ bounded in norm, for $t \geqslant 0$.

$$\dot{y}(t) = \sum_{i=1}^{k} A_i(t)\, y(t - r_i(t) - \tau_i(t, y_t)) \qquad \text{(Perturbed Equation)}, \tag{4.12}$$

where each $\tau_i(t, \phi)$ is a continuous functional from $R \times C[-q, 0] \to R$, $q > r$, and we will need some additional conditions to assure the equation is a retarded equation with bounded lags.

Let

$$Y(t, \phi) = \sum_{i=1}^{k} A_i(t)\, [\phi(-r_i(t) - \tau_i(t, \phi)) - \phi(-r_i(t))].$$

Clearly $| Y(t, \phi)| \leqslant 2H \| \phi \|$, where

$$H = \max_t \sum_{i=1}^{k} | A_i(t)| .$$

So $m(s) = 2Hs$. And if ϕ has a continuous derivative, it is also clear that

$$| Y(t, \phi)| \leqslant H\tau(t, \| \phi \|)\, \| \dot{\phi} \| ,$$

where $| \tau_i(t, \phi)| \leqslant \tau(t, \| \phi \|)$ for $1 \leqslant i \leqslant k$. Assuming $\tau(t, u)$ monotonic in u, let $N(t, u, s) = H\tau(t, u)\, s$. Then

$$N(t, u, Hu + m(u)) = H\tau(t, u)\, (3Hu) = 3H^2\tau(t, u)\, u.$$

Referring to (4.6), clearly

$$\psi(t) = 3H^2\tau(t, D), \qquad \text{for } t \geqslant 0, \quad | u | \leqslant D.$$

So (4.7) is satisfied if either $\tau(t, u) \to 0$ as $u \to 0$, uniformly for $t \geqslant 0$; or $\tau(t, D) \to 0$ as $t \to \infty$; or $\int_0^\infty \tau(t, D)\, dt < \infty$, for under these assumptions, (4.8), (4.9), or (4.10), respectively, are valid.

In addition, we must require that for some $q > r$, $0 \leqslant r_i(t) + \tau_i(t, \phi) \leqslant q$ for $| \phi | \leqslant D$. If $\tau(t, u) \to 0$ as $u \to 0$, uniformly for $t \geqslant 0$, this can be obtained for D sufficiently small. In the other two cases above, this becomes an additional assumption.

The foregoing is summarized in the following.

THEOREM 4.3. *Consider*

$$\dot{y}(t) = \sum_{i=1}^{k} A_i(t)\, y(t - r_i(t) - \tau_i(t, y_t)). \tag{4.12}$$

Assume the unperturbed equation ($\tau_i \equiv 0$) *is exponentially stable,* $0 \leqslant r_i(t) \leqslant r$, r_i, A_i *are continuous, and* $\sum_{i=1}^{k} | A_i(t)| \leqslant H$, *for* $t \geqslant 0$, *and constants r and H. Assume* $\tau(t, u)$ *is continuous in t and u, monotonic in u, and* $| \tau_i(t, \phi)| \leqslant \tau(t, \| \phi \|)$ *for* $\| \phi \| \leqslant B$. *Assume there is a constant* $D < B$ *and* $q > r$ *such that* $0 \leqslant r_i(t) + \tau_i(t, \phi) \leqslant q$ *for* $\| \phi \| \leqslant D$, $t \geqslant 0$. *Then the origin in* (4.12) *is exponentially stable, if* τ *satisfies any of the following.*

$$\tau(t, u) \to 0 \quad \text{as} \quad u \to 0, \quad \textit{uniformily in } t \textit{ for } t \geqslant 0, \tag{4.13}$$

$$\tau(t, D) \to 0 \quad \text{as} \quad t \to \infty, \tag{4.14}$$

$$\int_0^\infty \tau(t, D)\, dt < \infty. \tag{4.15}$$

Theorem 4.3 includes some examples in Grossman and Yorke [5] on exponential stability for perturbed delay equations.

The perturbations in the lag considered thus far are bounded lags which become small either as $\| \phi \| \to 0$ or $t \to \infty$. The method of approach used is sufficiently simple that more general lags are allowed. Consider the following.

THEOREM 4.4. *Consider* 4.12

$$\dot{y}(t) = \sum_{i=1}^{k} A_i(t)\, y(t - r_i(t) - \tau_i(t, y_t)).$$

Assume the unperturbed equation ($\tau_i \equiv 0$) *is exponentially stable, i.e.,* $| y_t(t_0, \phi)\| \leqslant K e^{-\sigma(t-t_0)} \| \phi \|$, *and* $0 \leqslant r_i(t) \leqslant r$, r_i, A_i *are continuous, and* $\sum_{i=1}^{k} | A_i(t)| \leqslant H$, *for* $t \geqslant 0$ *and constants r, H. Assume* $\tau(t, u)$ *is continuous in t, and u, monotonic in u, and* $0 \leqslant \tau_i(t, \phi) \leqslant \tau(t, \| \phi \|)$ *for* $\| \phi \| \leqslant B$, $1 \leqslant i \leqslant k$. *Given* $q > r$, *assume there exists constants D, k, a, b,* $a > 0$, $k < q - r$, *such that for* $0 \leqslant u \leqslant D$,

$$\tau(t, u) \leqslant k + u^a t^b, \tag{4.16}$$

and, given $\epsilon > 0$, *there exists* $\delta > 0$ *and* $T \geqslant 0$ *such that for* $0 \leqslant u \leqslant \delta$, $t \geqslant T$,

$$\tau(t, u) \leqslant \epsilon + u^a t^b. \tag{4.17}$$

Then given $\gamma < 1$, *there exists a constant* K_4 *and a continuous function* $\rho(t)$, *such that if* $\| \phi \| \leqslant \rho(t_0)$, *and* y_t *is a solution of* (4.12) *with* $y_{t_0} = \phi$, *then*

$$\| y_t \| \leqslant K_4 e^{-(1-\gamma)\sigma(t-t_0)} \| \phi \|, \qquad \textit{for } t \geqslant t_0 .$$

Proof. $\| y_t \|$ is dominated by $u(t)$, the maximal solution of

$$\dot{u} \leqslant -\sigma u + 2K e^{\sigma q} H u, \qquad \text{for } t_0 \leqslant t \leqslant t_0 + q, \tag{4.18}$$

and

$$\dot{u} \leqslant -\sigma u + 3Ke^{\sigma q}H^2\tau(t, u)\, u, \qquad \text{for } t \geqslant t_0 + q, \tag{4.19}$$

with

$$u(t_0) = W(t_0, \phi).$$

Of course, one must note in this case that (4.12) is meaningful only for those solutions y_t such that $\tau(t, \| y_t \|) \leqslant q - r$. So u dominates solutions of (4.12) only under these additional restrictions.

Choose $\gamma < 1$, and $t_0 \geqslant 0$. Given K, σ above, let $K_1 = Ke^{\sigma q}$. Choose ϵ in (4.17) so that $\epsilon \cdot 3K_1H^2 < \gamma\sigma/2$. This determines δ and T. Equation (4.18) shows that $u(t_0 + q)$ may be made as small as desired by taking $u(t_0)$ sufficiently small. If $t_0 + q < T$, then, from (4.16), for $| u | \leqslant \min(D, 1)$, (4.19) becomes

$$\dot{u} = (-\sigma + 3K_1H^2(k + T^b))\, u, \qquad t_0 + q \leqslant t \leqslant T. \tag{4.20}$$

And again, from (4.20) it follows that $u(T)$ may be made as small as desired by taking $u(t_0 + q)$ sufficiently small. So for $t \geqslant \max(T, t_0 + q)$, (4.19) becomes

$$\dot{u} = (-\sigma + 3K_1H^2\epsilon + 3K_1H^2u^a t^b)\, u. \tag{4.21}$$

But for $u(T)$ small enough, it is known (see [4, pp. 318–319]), that the maximal solution $u(t)$ of (4.21) satisfies

$$| u(t)| \leqslant K_3 e^{-(1-\gamma)\sigma(t-T)}\, | u(t)| , \qquad \text{for } t \geqslant T,$$

and some constant K_3. It follows that if $\| \phi \| < \rho(t_0)$, then $u(t)$, the maximal solution of (4.18) and (4.19), with $u(t_0) = W(t_0, \phi)$ satisfies

$$| u(t)| \leqslant K_4^{-(1-\gamma)\sigma(t-t_0)}\, \| \phi \| , \qquad \text{for } t \geqslant t_0 \text{, for some } K_4 > 0. \tag{4.22}$$

Further, $\rho(t_0)$ may be further restricted so that

$$\rho^a K_4{}^a e^{-a(1-\gamma)\sigma(t-t_0)} t^b \leqslant q - k - r, \qquad \text{for } t \geqslant t_0 . \tag{4.23}$$

Then $\| y_t \| \leqslant u(t)$ allows the conclusions that by (4.22), zero is asymptotically stable in (4.12), and from (4.23), the lag in (4.12) is restricted to the interval $[-q, 0]$ for those solutions whose initial condition is bounded in norm by $\rho(t_0)$. This completes the proof of Theorem 4.4.

As a concrete example of the above theorem, note that for $| b | < a$, the origin is exponentially stable for any lag r in

$$\dot{y}(t) = -ay(t) + by(t - r);$$

see [2, pp. 54–55].

Theorem 4.3 allows the conclusion that the origin is still exponentially stable for the following equations

$$\dot{y}(t) = -ay(t) + by(t - r - y^2(t)),$$

$$\dot{y}(t) = -ay(t) + by\left(t - r - \frac{\sin^2 t}{t^2 + 1}\right),$$

$$\dot{y}(t) = -ay(t) + by(t - r - \eta(t)),$$

where $0 \leqslant \eta(t) \leqslant c$, and $\int_0^\infty \eta(t)\,dt < \infty$, and η need not approach zero as $t \to \infty$.

Theorem 4.4 allows a similar conclusion for

$$\dot{y}(t) = -ay(t) + by(t - r - t^2 y(t)),$$

although in this case the stability is not uniform.

In the case where the unperturbed equation is uniformly stable, i.e., Eq. (4.5) dominates solutions, the same analysis as above leads to the following.

THEOREM 4.4. *Consider* 4.12

$$\dot{y}(t) = \sum_{i=1}^{k} A_i(t)\, y(t - r_i(t) - \tau_i(t, y_t)).$$

Assume the unperturbed equation ($\tau_i \equiv 0$) *is uniformly stable*, $0 \leqslant r_i(t) \leqslant r$, r_i, A_i *are continuous, and* $\sum_{i=1}^{k} |A_i(t)| \leqslant H$ *for* $t \geqslant 0$, *and some constants* r *and* H. *Assume there are constants* $q > r$, *and* $D < B$ *such that* $0 \leqslant \tau_i(t, \phi) + r_i(t) \leqslant q$ *for* $t \geqslant 0$, $\phi \in C[-q, 0]$, $\|\phi\| \leqslant D$. *Let* $\tau(t, u)$ *be continuous in* t *and* u, *monotonic in* u, *and* $|\tau_i(t, \phi)| \leqslant \tau(t, \|\phi\|)$, *for* $\|\phi\| \leqslant D$, $1 \leqslant i \leqslant k$. *If* $\int_0^\infty \tau(t, D)\,dt < \infty$, *then the origin in* (4.12) *is still uniformly stable.*

The proof consists simply of noting that now solutions of (4.12) are dominated by solutions of the equation

$$\dot{u}(t) = 3KH^2 \tau(t, u)\, u,$$

and solutions to this are clearly uniformly bounded for all $t \geqslant 0$.

An example of a nonlinear system is given by

$$\dot{y}(t) = \sum_{i=1}^{k} a_i(t, y(t))\, y(t - r_i(t)), \tag{4.24}$$

where we assume the origin is exponentially stable, i.e., (2.2) is satisfied, with

$\alpha(t) = \sigma t$, $\sigma > 0$, $K(t) = K > 0$. Assume further $r_i(t)$, $a_i(t, y)$ are continuous in t, y, a_i is lipschitzian on y with Lipschitz constant L_i, and $0 \leqslant r_i(t) \leqslant r$,

$$\sum_{i=1}^{k} | a_i(t, y)| \leqslant H(t, | y |) \qquad \text{for } t \geqslant 0, | y | \leqslant B.$$

Here $H(t, u)$ is continuous and monotonic in t and u.

Sufficient conditions on the a_i and r_i for exponential stability appear in the literature; see [2] or [6], for examples. One can readily verify that if

$$\| y_t(t_0, \phi)\| \leqslant K e^{-\sigma(t-t_0)} \| \phi \|,$$

then the conditions of Lemma 2.4 and Lemma 2.5 are satisfied for $T = (\log K)/k\sigma + q/k$ and the Lipschitz constant for $V(t, \phi)$ and $W(t, \phi)$ is $K_2 = e^{(L+(1-k)\sigma)T}$, where $L = \sum_{i=1}^{k} L_i$.

If we consider the perturbed equation

$$\dot{y}(t) = \sum_{i=1}^{k} a_i(t, y(t))\, y(t - r_i(t) - \tau_i(t, y_t)), \tag{4.25}$$

where as before $0 \leqslant r_i(t) + \tau_i(t, \phi) \leqslant q$, for some $q > r$ and $\| \phi \| \leqslant D \leqslant B$, with $| \tau_i(t, \phi)| \leqslant \tau(t, \| \phi \|)$ for all i, the dominating equation in this case, a specialization of (3.14), is

$$\dot{u} = -(1 - k)\, \sigma u + 3K_2 H^2(t, u)\, \tau(t, u)\, u. \tag{4.26}$$

And accordingly, the role of the function $\psi(t)$ in the previous analysis is played by $3K_2H^2(t, u)\, \tau(t, u)$ here. If one assumes $H(t, u)$ is uniformly bounded for $t \geqslant 0$, $| u | \leqslant B$, this reduces to the cases considered above. This case includes some results of Stephan [6].

The boundedness of H is clearly a reasonable hypothesis, but there are evidently other assumptions one could discuss in analyzing (4.26). These will not be discussed further.

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